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Stochastic collapse and decoherence of a non-dissipative forced harmonic oscillator

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Abstract

Careful monitoring of harmonically bound (or as a limiting case, free) masses is the basis of current and future gravitational wave detectors, and of nanomechanical devices designed to access the quantum regime. We analyse the effects of stochastic localization models for state vector reduction, and of related models for environmental decoherence, on such systems, focusing our analysis on the non-dissipative forced harmonic oscillator and its free mass limit. We derive an explicit formula for the time evolution of the expectation of a general operator in the presence of stochastic reduction or environmentally induced decoherence, for both the non-dissipative harmonic oscillator and the free mass. In the case of the oscillator, we also give a formula for the time evolution of the matrix element of the stochastic expectation density matrix between general coherent states. We show that the stochastic expectation of the variance of a Hermitian operator in any unraveling of the stochastic process is bounded by the variance computed from the stochastic expectation of the density matrix, and we develop a formal perturbation theory for calculating expectation values of operators within any unraveling. Applying our results to current gravitational wave interferometer detectors and nanomechanical systems, we conclude that the deviations from quantum mechanics predicted by the continuous spontaneous localization (CSL) model of state vector reduction are at least five orders of magnitude below the relevant standard quantum limits for these experiments. The proposed LISA gravitational wave detector will be two orders of magnitude away from the capability of observing an effect.

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1. Introduction

Testing whether quantum mechanics is an exactly correct theory, or is an approximate theory from which there are small deviations, is a subject of current theoretical and experimental interest. Significant bounds have been set [1] on deterministic, nonlinear modifications of the Schrödinger equation [2], and such modifications are also theoretically disfavoured because they have been shown [3] to lead to the possibility of superluminal communication. On the other hand, stochastic modifications to the Schrödinger equation have been extensively studied [4] as a way of resolving the measurement problem in quantum mechanics, and are known to be theoretically viable. This raises the question of what bounds on the stochasticity parameters are set by current experiments and what degree of refinement of current experiments will be needed to confront, and thus verify or falsify, the stochastic models.

The most extensively studied stochastic models are those based on the concept of localization [4, 5], in which a stochastic Brownian motion couples to the system centre-of-mass degree of freedom. Weak bounds on the stochasticity parameters for this type of model can already be set [6] from experiments [7] observing fullerene diffraction, and stronger (but far from definitive) bounds will be set [8] by a recently proposed experiment [9] that aims to coherently superimpose spatially displaced states of a small mirror attached to a cantilever. Our aim in this paper is to analyse the effects of stochastic localization on another class of precision experiments, involving the careful monitoring of massive objects in gravitational wave detectors, and of microscopic oscillating beams in nanomechanical experiments. To this end, we analyse the stochastic Schrödinger equation for a non-dissipative forced harmonic oscillator, focusing particular attention on the effects of the stochasticity terms on the quantum non-demolition variables of the oscillator. We also derive analogous formulae for the limiting case of a free mass, correcting a factor of 2 error in previous formulae given in the CSL literature. Because the stochastic expectation of the density matrix in the localization model obeys a differential equation used as a model for environmental decoherence, our results are also relevant to the study of decoherence effects on both the forced oscillator and free mass systems. Analysing various experiments using our results, we conclude that for the parameters of current gravitational wave detectors and nanomechanical beams, only weak bounds will be set on the CSL model stochasticity parameters. The proposed LISA gravitational wave detector should do better, but is still not expected to see an effect.

This paper is organized as follows. In section 2 we give the basic stochastic Schrödinger equation to be analysed, the corresponding pure state density matrix equation, and the simpler equation for the stochastic expectation of the density matrix (which is the usual mixed state density matrix). The latter equation, we note, is also used as a model for environmental decoherence effects, and so its solution is of particular interest. We also review briefly the basic ideas of quantum non-demolition measurements, leading to the identification of the non-demolition variables of the forced harmonic oscillator. In section 3 we give results for the time evolution of expectations of the non-demolition and other low-order variables of the forced oscillator. For comparison with the zero frequency limit of the oscillator, we give in section 4 analogous results for a free mass, rederiving (and correcting) results already in the literature. In section 5 we give formulae for the time evolution of stochastic expectations of general operators for the forced non-dissipative oscillator and for its free mass limit, and additionally derive a formula for transition amplitudes of the oscillator, giving results that also apply to environmental decoherence effects. In section 6 we consider stochastic fluctuations, and show that expectations of variances of observables can be bounded using our earlier calculations proceeding from the expectation of the density matrix. In section 7 we set up a formal perturbative procedure for calculating stochastic fluctuation effects, and use the leading order results to interpret the inequality derived in section 6. Finally, in section 8, we apply our results to make estimates for the effects of CSL models in gravitational wave detection and nanomechanical resonator experiments. In appendix A we review some Itô calculus formulae, and in appendix B we relate the formalism used in the text to the Lindblad density matrix evolution equation.

2. Basic formalism: one-dimensional oscillator

We start our analysis by considering a massive harmonic oscillator in one dimension, which in the three-dimensional case will describe the dynamics of one centre-of-mass degree of freedom. The oscillator Hamiltonian is taken as

$$H = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right) + d(t)a^\dagger + \overline{d}(t)a, \tag{1}$$

with ω the oscillator angular frequency, $d(t)$ a complex c-number driving term and $a, a^\dagger$ annihilation and creation operators respectively obeying $[a, a^\dagger] = 1$. These operators are related to the oscillator mass m, coordinate q and momentum p, by

$$\begin{aligned} a &= (m\omega/2\hbar)^{\frac{1}{2}}(q + \mathrm{i}p/m\omega), \\ a^\dagger &= (m\omega/2\hbar)^{\frac{1}{2}}(q - \mathrm{i}p/m\omega), \\ q &= \sigma(a + a^\dagger), \qquad \sigma = (\hbar/2m\omega)^{\frac{1}{2}} \end{aligned} \tag{2a}$$

and the number of quanta N in the oscillator is given by

$$N = a^\dagger a. \tag{2b}$$

Discussions of quantum non-demolition experiments involving oscillators [10] also introduce the quantities

$$X_1 = q\cos\omega t - (p/m\omega)\sin\omega t, \qquad X_2 = q\sin\omega t + (p/m\omega)\cos\omega t, \tag{3a}$$

from which one easily finds

$$\begin{aligned} q + \mathrm{i}p/m\omega &= (X_1 + \mathrm{i}X_2)\,\mathrm{e}^{-\mathrm{i}\omega t}, \\ q - \mathrm{i}p/m\omega &= (X_1 - \mathrm{i}X_2)\,\mathrm{e}^{\mathrm{i}\omega t}, \\ X_1 &= \sigma(a\,\mathrm{e}^{\mathrm{i}\omega t} + a^\dagger\,\mathrm{e}^{-\mathrm{i}\omega t}), \\ X_2 &= -\mathrm{i}\sigma(a\,\mathrm{e}^{\mathrm{i}\omega t} - a^\dagger\,\mathrm{e}^{-\mathrm{i}\omega t}). \end{aligned} \tag{3b}$$

Hence $X_{1,2}$ are quantum mechanical analogues of the classical amplitude of the oscillator, and when the external driving term $d(t)$ is zero they are conserved, as is the occupation number N. Because these quantities are constants of the motion in the absence of external forces, their measurements, while introducing uncertainties into the conjugate variables (which are the phase ϕ in the case of N, X_2 in the case of X_1 and X_1 in the case of X_2), do not feed the uncertainties in the conjugate variables back into the time evolution of the measured variable. Hence the variables N, X_1 and X_2 can in principle be measured to an accuracy not limited by the uncertainty principle, and are called 'quantum non-demolition' variables.

Letting $|\psi_t\rangle$ be the oscillator wavefunction at time t, the standard Schrödinger equation is

$$\mathrm{d}|\psi_t\rangle = -(\mathrm{i}/\hbar)H\,\mathrm{d}t|\psi_t\rangle. \tag{4a}$$

We shall be interested in this paper in a class of models [11] for state vector reduction, which modify equation (4a) by adding stochastic terms to the Schrödinger equation. Specifically, we shall consider the evolution equation

$$\mathrm{d}|\psi_t\rangle = \left[-\frac{\mathrm{i}}{\hbar}H\,\mathrm{d}t + \sqrt{\eta}(q - \langle q\rangle)\,\mathrm{d}W_t - \frac{\eta}{2}(q - \langle q\rangle)^2\,\mathrm{d}t\right]|\psi_t\rangle, \tag{4b}$$

where H is given by equation (1) and $\langle q\rangle \equiv \langle\psi_t|q|\psi_t\rangle$ is the quantum mechanical expectation of the position operator q of the oscillator. Introducing the pure state density matrix

$\hat{\rho}(t) = |\psi_t\rangle\langle\psi_t|$, we can also write $\langle q\rangle = \mathrm{Tr}\, q\hat{\rho}(t)$. The stochastic dynamics is governed by a standard Wiener processes W_t, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Using the rules of the Itô calculus (see appendix A), the density matrix evolution corresponding to equation (4*b*) is

$$\mathrm{d}\hat{\rho} = -\frac{\mathrm{i}}{\hbar}[H, \hat{\rho}]\,\mathrm{d}t - \frac{1}{2}\eta[q, [q, \hat{\rho}]]\,\mathrm{d}t + \sqrt{\eta}[\hat{\rho}, [\hat{\rho}, q]]\,\mathrm{d}W_t. \tag{5a}$$

Since this evolution equation obeys $\{\mathrm{d}\hat{\rho}, \hat{\rho}\} + (\mathrm{d}\hat{\rho})^2 = \mathrm{d}\hat{\rho}$, it preserves the pure state condition $\hat{\rho}^2 = \hat{\rho}$. When statistics are accumulated by averaging many runs of an experiment, the relevant density matrix in the stochastic case is the ensemble expectation $\rho = E[\hat{\rho}]$, giving the mixed state density matrix which obeys the ordinary differential equation

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\frac{\mathrm{i}}{\hbar}[H, \rho] - \frac{1}{2}\eta[q, [q, \rho]] \\ &= -\frac{\mathrm{i}}{\hbar}[H, \rho] - \frac{1}{2}\eta\sigma^2[a + a^\dagger, [a + a^\dagger, \rho]].\end{aligned} \tag{5b}$$

This equation is of particular interest because (with a different value of the parameter η) it is also used [12] as a simple model for environmental decoherence effects. The calculations of this paper focus on analysing equations (5*a*) and (5*b*) for the forced harmonic oscillator Hamiltonian of equation (1).

3. Stochastic expectations of oscillator observables

We begin by considering the evolution equation of equation (5*b*) for the mixed state density matrix ρ. Letting B be any time-independent operator, let us denote by $\langle\langle B\rangle\rangle$ the expectation computed with the mixed state density matrix,

$$\langle\langle B\rangle\rangle = \mathrm{Tr}\,\rho B. \tag{6a}$$

For the time evolution of this expectation, we then find

$$\begin{aligned}\frac{\mathrm{d}\langle\langle B\rangle\rangle}{\mathrm{d}t} &= \mathrm{Tr}\,\frac{\mathrm{d}\rho}{\mathrm{d}t}B \\ &= \mathrm{Tr}\,B\left(-\frac{\mathrm{i}}{\hbar}[H, \rho] - \frac{1}{2}\eta[q, [q, \rho]]\right) \\ &= \mathrm{Tr}\left(-\frac{\mathrm{i}}{\hbar}[B, H] - \frac{1}{2}\eta\sigma^2[[B, a + a^\dagger], a + a^\dagger]\right)\rho,\end{aligned} \tag{6b}$$

where we have made repeated use of cyclic permutation under the trace. Let us now make successively the choices $B = a, a^\dagger, aa, a^\dagger a^\dagger, a^\dagger a, aa^\dagger = 1 + a^\dagger a$, corresponding to all quantities linear and quadratic in the creation and annihilation operators. Then evaluating the single and double commutators in the final line of equation (6*b*), a simple calculation gives for the two linear operators,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\,\rho a = -\mathrm{i}\omega\,\mathrm{Tr}\,\rho a - \frac{\mathrm{i}}{\hbar}\,d(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\,\rho a^\dagger = \mathrm{i}\omega\,\mathrm{Tr}\,\rho a^\dagger + \frac{\mathrm{i}}{\hbar}\,\overline{d}(t), \tag{7a}$$

and for the four quadratic operators,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\rho aa &= -2\mathrm{i}\omega\operatorname{Tr}\rho aa - 2\frac{\mathrm{i}}{\hbar}d(t)\operatorname{Tr}\rho a - \eta\sigma^2,\\
\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\rho a^\dagger a^\dagger &= 2\mathrm{i}\omega\operatorname{Tr}\rho a^\dagger a^\dagger + 2\frac{\mathrm{i}}{\hbar}\overline{d}(t)\operatorname{Tr}\rho a^\dagger - \eta\sigma^2,\\
\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\rho a^\dagger a &= -\frac{\mathrm{i}}{\hbar}d(t)\operatorname{Tr}\rho a^\dagger + \frac{\mathrm{i}}{\hbar}\overline{d}(t)\operatorname{Tr}\rho a + \eta\sigma^2,\\
\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\rho aa^\dagger &= \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\rho a^\dagger a.
\end{aligned}
\tag{7b}
$$

These equations can be immediately integrated to give

$$
\begin{aligned}
\operatorname{Tr}\rho(t)a &= \mathrm{e}^{-\mathrm{i}\omega t}\left[\operatorname{Tr}\rho(0)a - \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}u\,\mathrm{d}(u)\,\mathrm{e}^{\mathrm{i}\omega u}\right],\\
\operatorname{Tr}\rho(t)a^\dagger &= \mathrm{e}^{\mathrm{i}\omega t}\left[\operatorname{Tr}\rho(0)a^\dagger + \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}u\,\overline{\mathrm{d}}(u)\,\mathrm{e}^{-\mathrm{i}\omega u}\right],
\end{aligned}
\tag{8a}
$$

for the linear operators and

$$
\begin{aligned}
\operatorname{Tr}\rho(t)aa &= \mathrm{e}^{-2\mathrm{i}\omega t}\left[\operatorname{Tr}\rho(0)aa - \int_0^t \mathrm{d}v\,\mathrm{e}^{2\mathrm{i}\omega v}\left(2\frac{\mathrm{i}}{\hbar}d(v)\operatorname{Tr}\rho(v)a + \eta\sigma^2\right)\right],\\
\operatorname{Tr}\rho(t)a^\dagger a^\dagger &= \mathrm{e}^{2\mathrm{i}\omega t}\left[\operatorname{Tr}\rho(0)a^\dagger a^\dagger + \int_0^t \mathrm{d}v\,\mathrm{e}^{-2\mathrm{i}\omega v}\left(2\frac{\mathrm{i}}{\hbar}\overline{d}(v)\operatorname{Tr}\rho(v)a^\dagger - \eta\sigma^2\right)\right],\\
\operatorname{Tr}\rho(t)a^\dagger a &= \operatorname{Tr}\rho(0)a^\dagger a - \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}v\left[d(v)\operatorname{Tr}\rho(v)a^\dagger - \overline{d}(v)\operatorname{Tr}\rho(v)a\right] + \eta\sigma^2 t,\\
\operatorname{Tr}\rho(t)aa^\dagger &= 1 + \operatorname{Tr}\rho(t)a^\dagger a,
\end{aligned}
\tag{8b}
$$

for the quadratic operators. We see that by substituting equation (8*a*) into equation (8*b*), we can reduce the expressions for the quadratic operators to quadratures. (Proceeding in a similar fashion, it is easy to see that given any polynomial $P(a, a^\dagger)$ of finite degree in the creation and annihilation operators, the expectation $\operatorname{Tr}\rho(t)P$ can be reduced to quadratures; for an explicit formula constructed by generating function methods, see section 5.)

Rather than exhibiting the full expressions for the expectations of the quadratic operators, we note that what we are most interested in is calculating the change in these quantities, denoted by δ, arising from the 'decoherence' term with coefficient η in equation (5*b*). From the fact that equation (8*a*) contains no terms proportional to η, we see that there are no stochastic (or decoherence) effects on the linear operators,

$$
\delta\operatorname{Tr}\rho(t)a = 0, \qquad \delta\operatorname{Tr}\rho(t)a^\dagger = 0,
\tag{9a}
$$

while the effect of the η term in equation (5*b*) on the quadratic operators is simply given by

$$
\begin{aligned}
\delta\operatorname{Tr}\rho(t)aa &= -\frac{\eta\sigma^2}{\omega}\mathrm{e}^{-\mathrm{i}\omega t}\sin\omega t,\\
\delta\operatorname{Tr}\rho(t)a^\dagger a^\dagger &= -\frac{\eta\sigma^2}{\omega}\mathrm{e}^{\mathrm{i}\omega t}\sin\omega t,\\
\delta\operatorname{Tr}\rho(t)a^\dagger a &= \delta\operatorname{Tr}\rho(t)aa^\dagger = \eta\sigma^2 t.
\end{aligned}
\tag{9b}
$$

Using the definitions of $X_{1,2}$ given in equations (3a) and (3b), we correspondingly find that

$$\begin{aligned}
&\delta\,\mathrm{Tr}\,\rho(t)X_1 = \delta\,\mathrm{Tr}\,\rho(t)X_2 = 0,\\
&\delta\,\mathrm{Tr}\,\rho(t)X_1^2 = 2\eta\sigma^4\left(t - \frac{\sin 2\omega t}{2\omega}\right),\\
&\delta\,\mathrm{Tr}\,\rho(t)X_2^2 = 2\eta\sigma^4\left(t + \frac{\sin 2\omega t}{2\omega}\right),\\
&\delta\,\mathrm{Tr}\,\rho(t)(X_1X_2 + X_2X_1) = -\frac{4\eta\sigma^4}{\omega}\sin^2\omega t,\\
&\delta\,\mathrm{Tr}\,\rho(t)[X_1, X_2] = 0.
\end{aligned} \tag{10}$$

We note that these formulae are exact (not just approximations to first order in η), since for all the operators B considered above, we have

$$\mathrm{Tr}\,\rho(t)B = \mathrm{Tr}\,\rho(t)B|_{\eta=0} + \delta\,\mathrm{Tr}\,\rho(t)B. \tag{11}$$

4. The free mass limit

According to equation (9b), the oscillator occupation number $N = a^\dagger a$ contains a term that grows linearly in time as $\eta\sigma^2 t$. Since the occupation number contribution to the oscillator energy of equation (1) is $\hbar\omega N$, and since $\sigma^2 = \hbar/2m\omega$ from equation (2a), the oscillator energy contains a term that grows linearly in time as

$$\delta E = \delta\,\mathrm{Tr}\,\rho(t)H = \frac{\eta\hbar^2 t}{2m}. \tag{12}$$

Because this formula is independent of the oscillator frequency ω, it should also correspond to the energy increase of an unbound mass m arising from the η term in equation (5b). This can be calculated directly as follows. For an unbound mass in one dimension, the Hamiltonian is $H = p^2/2m$, and the density matrix evolution is given by the first line of equation (5b). So we have, by the same reasoning that led to equation (6b),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\delta\,\mathrm{Tr}\,\rho H &= -\mathrm{Tr}\,\frac{1}{4m}\eta[[p^2, q], q]\rho\\
&= -\mathrm{Tr}\,\frac{1}{4m}\eta[-2\mathrm{i}\hbar p, q]\rho = \eta\hbar^2/2m,
\end{aligned} \tag{13a}$$

giving

$$\delta\,\mathrm{Tr}\,\rho H = \eta\hbar^2 t/2m \tag{13b}$$

in agreement with the result calculated for the oscillator. This result is a factor of 2 larger than the one quoted in the CSL literature [13]; for instance, Taylor expansion of equation (3.36) of Ghirardi, Pearle and Rimini (GPR) shows that for a uniform cube, their $\gamma\delta_i$ is the same as the parameter η used here, and so their formula of equation (3.38c), which states that $\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle P_i^2\rangle\rangle = \frac{1}{2}\gamma\delta_i\hbar^2$, would correspond to $\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle H\rangle\rangle = \eta\hbar^2/4m$, in disagreement with our result of equation (13a) and with the oscillator calculation of the preceding section. This error propagates through equations (3.41a) to (3.41c) of GPR, all of which are a factor of 2 too small. Thus, in our notation, their results should read

$$\begin{aligned}
&\delta\,\mathrm{Tr}\,\rho p^2 = \eta\hbar^2 t,\\
&\delta\,\mathrm{Tr}\,\rho(pq + qp) = \eta\hbar^2 t^2/m,\\
&\delta\,\mathrm{Tr}\,\rho q^2 = \eta\hbar^2 t^3/(3m^2).
\end{aligned} \tag{13c}$$

A rederivation of the second and third lines of equation (13c) will be given in the next section. (These equations were first given, with a different identification of the proportionality constant η, in the GRW model [11]. Philip Pearle has rechecked the calculations in the paper of GPR, and finds that a factor of 2 error was made in going from their equation (3.36) to their equation (3.38c); when corrected, their equations agree with our results of equation (13c) above.)

5. Exact general formulae for decoherence effects on a forced non-dissipative harmonic oscillator and on a free mass

We have seen in equations (8a) and (8b) that the double expectations (stochastic and quantum, as defined in equation (6a)) of low order polynomials in the oscillator creation and annihilation operators can be reduced to quadratures. To show that this is a general result, let us consider the generating function

$$\begin{aligned} K_{\alpha\beta}(t) &= \mathrm{Tr}(\exp(\alpha a^\dagger \,\mathrm{e}^{-\mathrm{i}\omega t})\exp(\beta a\,\mathrm{e}^{\mathrm{i}\omega t})\rho(t)) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^n}{n!}\frac{\beta^m}{m!}\,\mathrm{e}^{\mathrm{i}\omega t(m-n)}\,\mathrm{Tr}(a^\dagger)^n a^m \rho(t), \end{aligned} \tag{14a}$$

from which one can extract the expectations of arbitrary normal ordered operators formed from a and $a^\dagger$. To proceed, we shall need the generalization of equation (6b) to the case when the operator B has an explicit time dependence, which reads

$$\begin{aligned} \frac{\mathrm{d}\langle\langle B\rangle\rangle}{\mathrm{d}t} &= \mathrm{Tr}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}B + \rho\frac{\partial B}{\partial t}\right) \\ &= \mathrm{Tr}\left(\frac{\partial B}{\partial t} - \frac{\mathrm{i}}{\hbar}[B, H] - \frac{1}{2}\eta\sigma^2[[B, a + a^\dagger], a + a^\dagger]\right)\rho. \end{aligned} \tag{14b}$$

Applying this formula to equation (14a), with $B = \exp(\alpha a^\dagger \,\mathrm{e}^{-\mathrm{i}\omega t})\exp(\beta a\,\mathrm{e}^{\mathrm{i}\omega t})$, the explicit time derivative on the right cancels the commutator term involving the free Hamiltonian $\hbar\omega a^\dagger a$ (this is why we included an explicit time dependence in the definition of the generating function), leaving the simple differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}K_{\alpha\beta}(t) = \left[\frac{\mathrm{i}}{\hbar}(\alpha\,\mathrm{e}^{-\mathrm{i}\omega t}\overline{d}(t) - \beta\,\mathrm{e}^{\mathrm{i}\omega t}\,d(t)) - \frac{1}{2}\eta\sigma^2(\alpha\,\mathrm{e}^{-\mathrm{i}\omega t} - \beta\,\mathrm{e}^{\mathrm{i}\omega t})^2\right]K_{\alpha\beta}(t). \tag{15}$$

Defining

$$D(t) \equiv \int_0^t \mathrm{d}u\,\mathrm{e}^{\mathrm{i}\omega u}\,\mathrm{d}(u), \qquad \overline{D}(t) \equiv \int_0^t \mathrm{d}u\,\mathrm{e}^{-\mathrm{i}\omega u}\,\overline{d}(u), \tag{16a}$$

the integral of equation (15) takes the form

$$K_{\alpha\beta}(t) = \exp\left[\alpha\beta\eta\sigma^2 t - \frac{\eta\sigma^2}{2\omega}(\alpha^2\,\mathrm{e}^{-\mathrm{i}\omega t} + \beta^2\,\mathrm{e}^{\mathrm{i}\omega t})\sin\omega t + \frac{\mathrm{i}}{\hbar}(\alpha\overline{D}(t) - \beta D(t))\right]K_{\alpha\beta}(0), \tag{16b}$$

with

$$K_{\alpha\beta}(0) = \mathrm{Tr}(\mathrm{e}^{\alpha a^\dagger}\,\mathrm{e}^{\beta a}\rho(0)). \tag{16c}$$

Expanding this equation through second order in α and β, one can verify that it agrees with the formulae of equations (8a) and (8b), and so we have obtained the generalization of these expressions to arbitrary normal ordered monomials in the creation and annihilation operators. Thus expectations of operators with respect to the density matrix of the decoherent forced oscillator can be explicitly calculated in a closed form. As an example of particular interest, we note that the η-dependent terms with the dominant time dependence for large times can be read off from the power series expansion of the first factor on the right-hand side of equation (16*b*),

$$\exp(\alpha\beta\eta\sigma^2 t) = \sum_{n=0}^{\infty} \frac{(\alpha\beta\eta\sigma^2 t)^n}{n!}. \tag{16d}$$

Thus, the leading η dependence in $\mathrm{Tr}\,\rho(t)a^\dagger a$ at large times is $\eta\sigma^2 t$, in agreement with equation (8*b*), while the leading η dependence in $\mathrm{Tr}\,\rho(t)a^\dagger a^\dagger aa$ is $2\eta^2\sigma^4 t^2$. We will apply these results below to a discussion of the variance of N at large times.

The same strategy that we have just followed can be used to find a generating function for the expectations of general polynomials in the operators q and p in the free particle case. Here the Hamiltonian is $H = p^2/(2m)$, and the equation to be solved is

$$\begin{aligned}\frac{\mathrm{d}\langle\langle B\rangle\rangle}{\mathrm{d}t} &= \mathrm{Tr}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}B + \rho\frac{\partial B}{\partial t}\right) \\ &= \mathrm{Tr}\left(\frac{\partial B}{\partial t} - \frac{\mathrm{i}}{\hbar}[B, H] - \frac{1}{2}\eta[[B, q], q]\right)\rho. \end{aligned}\tag{17a}$$

We consider now the generating function defined by

$$K^f_{\alpha\beta} = \mathrm{Tr}\,B\rho(t) = \mathrm{Tr}[\exp(\alpha(q - tp/m))\exp(\beta p)\rho(t)]. \tag{17b}$$

Using the fact that

$$\exp(-(\mathrm{i}/\hbar)tp^2/(2m))q\exp((\mathrm{i}/\hbar)tp^2/(2m)) = q - tp/m, \tag{17c}$$

we see that the terms $\partial B/\partial t$ and $-(\mathrm{i}/\hbar)[B, H]$ in equation (17*a*) cancel, so we are left with

$$\frac{\mathrm{d}}{\mathrm{d}t}K^f_{\alpha\beta}(t) = \mathrm{Tr}\left(-\frac{1}{2}\eta[[B, q], q]\rho(t)\right). \tag{18a}$$

Using now the identity

$$\exp(\alpha(q - tp/m))\exp(\beta p) = \exp(\alpha q)\exp(p(\beta - \alpha t/m))\exp(\alpha^2 i\hbar t/(2m)), \tag{18b}$$

the right-hand side of equation (18*a*) is easily evaluated to give

$$\tfrac{1}{2}\eta\hbar^2(\beta - \alpha t/m)^2 K^f_{\alpha\beta}(t). \tag{18c}$$

Equations (18*a*) and (18*c*) now give a differential equation that can be immediately integrated, giving a result analogous in form to equation (16*b*),

$$K^f_{\alpha\beta}(t) = \exp\left[\tfrac{1}{6}\eta\hbar^2 t(3\beta^2 - 3\beta\alpha t/m + \alpha^2 t^2/m^2)\right]K^f_{\alpha\beta}(0), \tag{19a}$$

with

$$K^f_{\alpha\beta}(0) = \mathrm{Tr}(\exp(\alpha q)\exp(\beta p)\rho(0)). \tag{19b}$$

This equation gives a generating function from which the results of appendix E of Ghirardi, Rimini and Weber [11] and their extensions to higher order polynomials, can be readily extracted. In particular, expanding equation (19*a*) through second order in α and β, one gets for the leading η dependence of the expectations of quadratic polynomials in p and q the expressions given above in equation (13*c*).

Returning to the harmonic oscillator, the same methods can be applied to the generating function for general matrix elements of $\rho(t)$, although the results in this case are not so simple. Let us define the generating function

$$L_{\alpha\beta}(t) = \mathrm{Tr}(\exp(\alpha a^\dagger\,\mathrm{e}^{-\mathrm{i}\omega t})|0\rangle\langle 0|\exp(\beta a\,\mathrm{e}^{\mathrm{i}\omega t})\rho(t)), \tag{20a}$$

where $|0\rangle$ is the oscillator ground state obeying $a|0\rangle = \langle 0|a^\dagger = 0$. (With the inclusion of a normalization factor $\exp\left(-\frac{1}{2}(|\alpha|^2 + |\beta|^2)\right)$, this expression gives directly the matrix element of $\rho(t)$ between coherent states of the oscillator parametrized by α and β.) When we take the time derivative of this expression, and apply equation (14b), we now find that there are additional terms where an $a^\dagger$ multiplies $|0\rangle$ from the left, or an a multiplies $\langle 0|$ from the right. These can be converted to derivatives of $L_{\alpha\beta}$ with respect to the parameters α and β, and so we end up with the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\alpha\beta}(t) = \left\{ \frac{\mathrm{i}}{\hbar} \left[\left(\alpha - \frac{\partial}{\partial\beta}\right) \mathrm{e}^{-\mathrm{i}\omega t}\, \overline{\mathrm{d}}(t) - \left(\beta - \frac{\partial}{\partial\alpha}\right) \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}(t) \right] - \frac{1}{2}\eta\sigma^2 \left[\left(\alpha - \frac{\partial}{\partial\beta}\right) \mathrm{e}^{-\mathrm{i}\omega t} - \left(\beta - \frac{\partial}{\partial\alpha}\right) \mathrm{e}^{\mathrm{i}\omega t} \right]^2 \right\} L_{\alpha\beta}(t), \tag{20b}$$

which corresponds to making the substitutions $\alpha \to \alpha - \frac{\partial}{\partial\beta}$, $\beta \to \beta - \frac{\partial}{\partial\alpha}$ in equation (15). Since the operators $\frac{\partial}{\partial\beta} - \alpha$ and $\frac{\partial}{\partial\alpha} - \beta$ commute with one another, this equation can be formally integrated without requiring a time ordered product. Using

$$\left(\frac{\partial}{\partial\alpha} - \beta\right) = \mathrm{e}^{\alpha\beta} \frac{\partial}{\partial\alpha}\, \mathrm{e}^{-\alpha\beta}, \qquad \left(\frac{\partial}{\partial\beta} - \alpha\right) = \mathrm{e}^{\alpha\beta} \frac{\partial}{\partial\beta}\, \mathrm{e}^{-\alpha\beta}, \tag{21a}$$

the result can be compactly written as

$$L_{\alpha\beta} = \mathrm{e}^{\alpha\beta} \exp\left[\eta\sigma^2 t \frac{\partial}{\partial\beta}\frac{\partial}{\partial\alpha} - \frac{1}{2}\eta\sigma^2 \frac{\sin\omega t}{\omega} \left(\mathrm{e}^{-i\omega t} \left(\frac{\partial}{\partial\beta}\right)^2 + \mathrm{e}^{\mathrm{i}\omega t} \left(\frac{\partial}{\partial\alpha}\right)^2 \right) \right] \mathrm{e}^{-\alpha\beta} L^0_{\alpha\beta}(t), \tag{21b}$$

with $L^0_{\alpha\beta}(t)$ the generating function in the absence of decoherence (that is, with $\eta = 0$), which is given by

$$L^0_{\alpha\beta}(t) = \exp\left(-\frac{1}{\hbar^2}|D(t)|^2 + \frac{\mathrm{i}}{\hbar}[\overline{D}(t)\alpha - D(t)\beta] \right) \mathrm{Tr}\, \mathrm{e}^{a^\dagger[\alpha + (\mathrm{i}/\hbar)D(t)]} |0\rangle\langle 0|\, \mathrm{e}^{a[\beta - (\mathrm{i}/\hbar)\overline{D}(t)]} \rho(0). \tag{21c}$$

An alternative form of this result is obtained by introducing the Fourier transform of $\mathrm{e}^{-\alpha\beta} L^0_{\alpha\beta}$ with respect to α and β,

$$\mathrm{e}^{-\alpha\beta} L^0_{\alpha\beta}(t) = \int \mathrm{d}p_\alpha\, \mathrm{d}p_\beta\, F(p_\alpha, p_\beta, t)\, \mathrm{e}^{\mathrm{i}\alpha p_\alpha + \mathrm{i}\beta p_\beta}, \tag{22a}$$

in terms of which equation (21b) takes the form

$$L_{\alpha\beta}(t) = \mathrm{e}^{\alpha\beta} \int \mathrm{d}p_\alpha\, \mathrm{d}p_\beta \exp\left[-\eta\sigma^2 t p_\alpha p_\beta + \frac{1}{2}\eta\sigma^2 \frac{\sin\omega t}{\omega} \left(\mathrm{e}^{-\mathrm{i}\omega t} p_\beta^2 + \mathrm{e}^{\mathrm{i}\omega t} p_\alpha^2 \right) \right] \times F(p_\alpha, p_\beta, t)\, \mathrm{e}^{\mathrm{i}\alpha p_\alpha + \mathrm{i}\beta p_\beta}. \tag{22b}$$

Thus, matrix elements of the density matrix for the decoherent forced oscillator can be explicitly (if formally) expressed in terms of matrix elements of the oscillator in the absence of decoherence.

We have seen that exact results can be obtained for a number of properties of the density matrix evolution equation of equation (5b). This might have been suspected from the fact that earlier work [8] has shown that this equation leads to an exactly solvable expression for the fringe visibility in a proposed mirror superposition experiment described by an oscillator Hamiltonian. More general density matrix evolution equations for a damped harmonic oscillator have been discussed in the literature [14]. When additional decoherence terms of the form $[c_1q + c_2p, [c_1q + c_2p, \rho]]$ (for general constants $c_{1,2}$) are added to the density matrix evolution equation for the forced oscillator, an explicit result for $K_{\alpha\beta}(t)$ generalizing equation (16*b*) can still be easily obtained. When dissipative terms proportional to a linear combination of $\mathrm{i}[q, \{p, \rho\}]$ and $\mathrm{i}[p, \{q, \rho\}]$, with $\{,\}$ the anticommutator, are added to the density matrix evolution equation, the differential equation for $K_{\alpha\beta}(t)$ contains terms involving $\partial/\partial\alpha$ and $\partial/\partial\beta$, and we then can no longer obtain an explicit formula for the expectation of the generating function analogous to equation (16*b*). Such dissipative terms are included in the evolution equations discussed in [14], where some exact results are obtained. We remark, however, that for mechanical or electrical systems with a very high quality factor Q, it can be a useful first approximation to neglect classical damping in studying stochastic reduction and decoherence effects, as done in the analysis of this paper. (For the benefit of the reader familiar with the Lindblad form of the density matrix evolution equation, we give in appendix B its relation to the commutator/anticommutator structures discussed here.)

6. Bounds on variances for unravelings

So far we have studied quantum expectations of physical quantities in the mixed state density matrix ρ obtained as the stochastic expectation of the pure state density matrix $\hat{\rho}$ that obeys equation (5*a*). In any given run of the physical process (or 'unraveling' in the stochastics literature parlance), the quantum expectation of a physical quantity represented by a non-stochastic operator B will be governed by $\mathrm{Tr}\,\hat{\rho}B$. As before, let us use the notation $\langle\cdots\rangle$ to denote expectations formed with respect to $\hat{\rho}$, and the notation $\langle\langle\cdots\rangle\rangle$ to denote expectations formed with respect to $\rho = E[\hat{\rho}]$. Then by linearity we evidently have

$$\langle\langle B\rangle\rangle = E[\langle B\rangle]. \tag{23}$$

We shall now show that the variances corresponding to the single and double averages are related by an inequality. Let

$$\langle(\Delta B)^2\rangle = \mathrm{Tr}\,\hat{\rho}(B - \mathrm{Tr}\,\hat{\rho}B)^2 = \mathrm{Tr}\,\hat{\rho}B^2 - (\mathrm{Tr}\,\hat{\rho}B)^2 \tag{24a}$$

be the squared variance of B formed with respect to $\hat{\rho}$, and

$$\langle\langle(\Delta B)^2\rangle\rangle = \mathrm{Tr}\,\rho(B - \mathrm{Tr}\,\rho B)^2 = \mathrm{Tr}\,\rho B^2 - (\mathrm{Tr}\,\rho B)^2 \tag{24b}$$

be the corresponding squared variance of B formed with respect to ρ. The first of these two squared variances fluctuates from unraveling to unraveling; taking its expectation over the stochastic process we have

$$\begin{aligned} E[\langle(\Delta B)^2\rangle] &= \mathrm{Tr}\,\rho B^2 - E[(\mathrm{Tr}\,\hat{\rho}B)^2] \\ &= \mathrm{Tr}\,\rho B^2 - (\mathrm{Tr}\,\rho B)^2 + C, \end{aligned} \tag{25a}$$

with C a correction term given by

$$\begin{aligned} C &= (\mathrm{Tr}\,E[\hat{\rho}]B)^2 - E[(\mathrm{Tr}\,\hat{\rho}B)^2] \\ &= -E[(\mathrm{Tr}\,\hat{\rho}B - \mathrm{Tr}\,E[\hat{\rho}]B)^2] \leqslant 0. \end{aligned} \tag{25b}$$

Hence we have obtained the inequality

$$E[\langle(\Delta B)^2\rangle] \leqslant \mathrm{Tr}\,\rho B^2 - (\mathrm{Tr}\,\rho B)^2 = \langle\langle(\Delta B)^2\rangle\rangle; \tag{26}$$

in other words, the squared variance formed from ρ gives an upper bound to the expectation of the squared variance formed from $\hat{\rho}$. These results, and those of section 3, can be used to calculate bounds on the expected variances $E[\langle(\Delta X_{1,2})^2\rangle]$. When the effects of the driving terms $\mathrm{d}(t), \overline{\mathrm{d}}(t)$ can be neglected (or at least remain bounded), we see, for example, that at large times we have from equation (10)

$$E[\langle(\Delta X_{1,2})^2\rangle] \leqslant \mathrm{Tr}\,\rho(t)X_{1,2}^2 - (\mathrm{Tr}\,\rho(t)X_{1,2})^2 \simeq 2\eta\sigma^4 t, \tag{27a}$$

giving a large time bound on the mean squared stochastic fluctuations of $X_{1,2}$. Similarly, we find (using the discussion following equation (16*d*)) that when the effects of the driving terms can be neglected, the leading large time variance of N is bounded by

$$\begin{aligned} E[\langle(\Delta N)^2\rangle] &\leqslant \mathrm{Tr}\,\rho(t)N^2 - (\mathrm{Tr}\,\rho(t)N)^2 = \mathrm{Tr}\,\rho(t)(a^\dagger a^\dagger aa + a^\dagger a) - (\mathrm{Tr}\,\rho(t)a^\dagger a)^2 \\ &\simeq 2\eta^2\sigma^4 t^2 - (\eta\sigma^2 t)^2 = \eta^2\sigma^4 t^2. \end{aligned} \tag{27b}$$

Thus the root mean square variance in N and the expectation of N have the same time rate of growth.

7. Perturbation analysis for stochastic fluctuations

We conclude our theoretical analysis by developing a formal perturbation theory for solving the evolution equation of equation (5*a*) for the pure state density matrix $\hat{\rho}$. Let $\rho^{(0)}$ obey the evolution equation

$$\mathrm{d}\rho^{(0)} = -\frac{i}{\hbar}[H, \rho^{(0)}]\,\mathrm{d}t, \tag{28a}$$

which holds when there are no stochastic terms, and let us expand the solution $\hat{\rho}$ of the corresponding stochastic equation as

$$\hat{\rho} = \rho^{(0)} + \sqrt{\eta}\hat{\rho}^{(1/2)} + \eta\hat{\rho}^{(1)} + \cdots. \tag{28b}$$

Inserting this expansion into equation (5*a*), and equating like powers of η on left and right, we get the following stochastic differential equations for $\hat{\rho}^{(1/2)}$ and $\hat{\rho}^{(1)}$:

$$\begin{aligned} \mathrm{d}\hat{\rho}^{(1/2)} &= -\frac{\mathrm{i}}{\hbar}[H, \hat{\rho}^{(1/2)}]\,\mathrm{d}t + \sigma[\rho^{(0)}, [\rho^{(0)}, a + a^\dagger]]\,\mathrm{d}W_t, \\ \mathrm{d}\hat{\rho}^{(1)} &= -\frac{\mathrm{i}}{\hbar}[H, \hat{\rho}^{(1)}]\,\mathrm{d}t - \frac{1}{2}\sigma^2[a + a^\dagger, [a + a^\dagger, \rho^{(0)}]]\,\mathrm{d}t \\ &\quad +\sigma\big([\rho^{(0)}, [\hat{\rho}^{(1/2)}, a + a^\dagger]] + [\hat{\rho}^{(1/2)}, [\rho^{(0)}, a + a^\dagger]]\big)\,\mathrm{d}W_t. \end{aligned} \tag{28c}$$

The first step in solving these equations is to eliminate the time evolution associated with the Hamiltonian term $\hbar\omega a^\dagger a$ by defining, for any operator B, an interaction picture operator B^I given by

$$B^I = \mathrm{e}^{\mathrm{i}\omega a^\dagger at} B\, \mathrm{e}^{-\mathrm{i}\omega a^\dagger at}, \tag{29a}$$

so that in particular

$$a^I = a\mathrm{e}^{-\mathrm{i}\omega t}, \qquad a^{I\dagger} = a^\dagger\, \mathrm{e}^{\mathrm{i}\omega t}. \tag{29b}$$

Then in the interaction picture, equations (28*a*) and (28*c*) become

$$\begin{aligned} \mathrm{d}\rho^{I(0)} &= -\frac{\mathrm{i}}{\hbar}[h^I, \rho^{I(0)}]\,\mathrm{d}t, \\ \mathrm{d}\hat{\rho}^{I(1/2)} &= -\frac{\mathrm{i}}{\hbar}[h^I, \hat{\rho}^{I(1/2)}]\,\mathrm{d}t + \sigma[\rho^{I(0)}, [\rho^{I(0)}, a\,\mathrm{e}^{-\mathrm{i}\omega t} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}]]\,\mathrm{d}W_t, \\ \mathrm{d}\hat{\rho}^{I(1)} &= -\frac{\mathrm{i}}{\hbar}[h^I, \hat{\rho}^{I(1)}]\,\mathrm{d}t - \frac{1}{2}\sigma^2[a\,\mathrm{e}^{-\mathrm{i}\omega t} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}, [a\,\mathrm{e}^{-\mathrm{i}\omega t} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}, \rho^{I(0)}]]\,\mathrm{d}t \\ &\quad + \sigma\big([\rho^{I(0)}, [\hat{\rho}^{I(1/2)}, a\,\mathrm{e}^{-\mathrm{i}\omega t} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}]] + [\hat{\rho}^{I(1/2)}, [\rho^{I(0)}, a\,\mathrm{e}^{-\mathrm{i}\omega t} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}]]\big)\,\mathrm{d}W_t. \end{aligned} \tag{29c}$$

Here $h^I = h^I(t)$ denotes the interaction picture form of the oscillator driving terms in the Hamiltonian,

$$h^I(t) = d(t)a^\dagger\,\mathrm{e}^{\mathrm{i}\omega t} + \overline{d}(t)a\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{29d}$$

We can now deal with the h^I term in the equations of motion by introducing an operator $U^I(t)$ that obeys the differential equation

$$\frac{\mathrm{d}U^I(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}h^I(t)U^I(t), \qquad \frac{\mathrm{d}U^I(t)^\dagger}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}U^I(t)^\dagger h^I(t), \tag{30a}$$

which, using the definitions of equation (16*a*), can be explicitly integrated to give

$$U^I(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}\overline{D}(t)a\right)\exp\left(-\frac{\mathrm{i}}{\hbar}D(t)a^\dagger\right)\exp\left(\frac{1}{\hbar^2}\int_0^t \mathrm{d}u\,\mathrm{d}(u)\,\mathrm{e}^{\mathrm{i}\omega u}\overline{D}(u)\right). \tag{30b}$$

We can now use U^I as an integrating factor to integrate equation (29*c*), giving finally explicit formulae for $\rho^{I(0)}$, $\hat{\rho}^{I(1/2)}$ and $\hat{\rho}^{I(1)}$,

$$\begin{aligned}
&\rho^{I(0)}(t) = U^I(t)\rho^{I(0)}(0)U^I(t)^\dagger,\\
&\hat{\rho}^{I(1/2)}(t) = \sigma\int_0^t U^I(t)U^I(s)^\dagger[\rho^{I(0)}(s),[\rho^{I(0)}(s), a\,\mathrm{e}^{-\mathrm{i}\omega s} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega s}]]U^I(s)U^I(t)^\dagger\,\mathrm{d}W_s,\\
&\hat{\rho}^{I(1)}(t) = -\frac{1}{2}\sigma^2\int_0^t U^I(t)U^I(s)^\dagger[a\,\mathrm{e}^{-\mathrm{i}\omega s} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega s}, [a\,\mathrm{e}^{-\mathrm{i}\omega s} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega s},\\
&\rho^{I(0)}(s)]]U^I(s)U^I(t)^\dagger\,\mathrm{d}s + \sigma\int_0^t U^I(t)U^I(s)^\dagger([\rho^{I(0)}(s),[\hat{\rho}^{I(1/2)}(s), a\,\mathrm{e}^{-\mathrm{i}\omega s} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega s}]]\\
&\qquad + [\hat{\rho}^{I(1/2)}(s),[\rho^{I(0)}(s), a\,\mathrm{e}^{-\mathrm{i}\omega s} + a^\dagger e^{\mathrm{i}\omega s}]])U^I(s)U^I(t)^\dagger\,\mathrm{d}W_s.
\end{aligned} \tag{31a}$$

These equations give terms in the expansion in powers of $\sqrt{\eta}$ of the interaction picture quantity $\hat{\rho}^I$; to transform back to the original Schrödinger picture, one uses the inverse of equation (29*a*),

$$B = \mathrm{e}^{-\mathrm{i}\omega a^\dagger a t}B^I\,\mathrm{e}^{\mathrm{i}\omega a^\dagger a t}, \tag{31b}$$

taking B to be successively $\hat{\rho}^{I(0,1/2,1)}$. From the results of this calculation, one can in principle compute the quantum expectations $\mathrm{Tr}\,\hat{\rho}B$ corresponding to different unravelings of the stochastic process.

We see from equation (31*a*) that the expression for $\hat{\rho}^{(1/2)}$ involves a stochastic integration over $\mathrm{d}W_s$ with a non-stochastic integrand, and so as expected we have $E[\hat{\rho}^{(1/2)}(t)] = 0$. Additionally, we note that the expression for $\hat{\rho}^{(1)}$ contains two integrals, an ordinary integral involving an integration over $\mathrm{d}s$, and a stochastic integral involving an integration over $\mathrm{d}W_s$. Since the integrand of the latter contains only stochastic quantities depending (through $\hat{\rho}^{I(1/2)}(s)$) on $\mathrm{d}W_t$ for $t \leqslant s$, the stochastic expectation of the integral over $\mathrm{d}W_s$ is zero, and so we have

$$\begin{aligned}
E[\hat{\rho}^{I(1)}(t)] = -\frac{1}{2}\sigma^2\int_0^t U^I(t)U^I(s)^\dagger[a\,\mathrm{e}^{-\mathrm{i}\omega s}\\
+ a^\dagger\,\mathrm{e}^{\mathrm{i}\omega s}, [a\,\mathrm{e}^{-\mathrm{i}\omega s} + a^\dagger\,\mathrm{e}^{\mathrm{i}\omega s}, \rho^{I(0)}(s)]]U^I(s)U^I(t)^\dagger\,\mathrm{d}s,
\end{aligned} \tag{32}$$

which writing $\rho^{I(1)}(s) = E[\hat{\rho}^{I(1)}(s)]$ gives the first term in the perturbation expansion for the ensemble expectation density matrix $\rho(t)$ obeying equation (5*b*).

Let us now use the results $E[\hat{\rho}^{(1/2)}(t)] = 0$ and $E[\hat{\rho}^{(1)}(t)] = \rho^{(1)}(t)$ to interpret the inequality derived in section 6. Inserting the expansion for $\hat{\rho}$ into the definition of equation (24*a*), we have

$$\langle(\Delta B)^2\rangle = \mathrm{Tr}(\rho^{(0)} + \sqrt{\eta}\hat{\rho}^{(1/2)} + \eta\hat{\rho}^{(1)})B^2 - (\mathrm{Tr}(\rho^{(0)} + \sqrt{\eta}\hat{\rho}^{(1/2)} + \eta\hat{\rho}^{(1)})B)^2. \tag{33a}$$

Taking now the expectation of this equation, we get for non-stochastic operators B,

$$E[\langle(\Delta B)^2\rangle] = \mathrm{Tr}(\rho^{(0)} + \eta\rho^{(1)})B^2 - (\mathrm{Tr}(\rho^{(0)} + \eta\rho^{(1)})B)^2 - \eta E[(\mathrm{Tr}\,\hat{\rho}^{(1/2)}B)^2] + \mathrm{O}(\eta^2). \tag{33b}$$

But comparing now with equation (24b), we see that this is just

$$E[\langle(\Delta B)^2\rangle] = \langle\!\langle(\Delta B)^2\rangle\!\rangle - \eta E[(\mathrm{Tr}\,\hat{\rho}^{(1/2)}B)^2] + \mathrm{O}(\eta^2), \tag{33c}$$

in agreement with the expansion of the inequality of equation (26) through terms of first order in η, and giving us insight into why the inequality takes this form. By writing equation (31a) for $\hat{\rho}^{(1/2)}$ in the form

$$\hat{\rho}^{(1/2)}(t) = \int_0^t P(s,t)\,\mathrm{d}W_s, \tag{34a}$$

with $P(s,t)$ denoting the integrand in equation (31a), and using the Itô isometry given in appendix A, the stochastic expectation in the final term in equation (33c) can be explicitly evaluated as an ordinary integral,

$$E[(\mathrm{Tr}\,\hat{\rho}^{(1/2)}B)^2] = \int_0^t \mathrm{d}s(\mathrm{Tr}\,P(s,t)B)^2. \tag{34b}$$

8. Estimates for gravitational wave detection and nanomechanical oscillator experiments

Let us now use the results of the preceding sections to make estimates for precision experiments involving monitoring of harmonically bound or free masses. We begin by collecting the relevant formulae. For the harmonic oscillator, we have seen in equation (9b) that the double expectation of the occupation number $N = a^\dagger a$ has a secular growth given by

$$\langle\langle N\rangle\rangle \simeq \eta\sigma^2 t. \tag{35a}$$

Since by our definition of equation (2a), $\sigma = (\hbar/2m\omega)^{\frac{1}{2}}$, we have [10] $\sigma = \Delta X_{\mathrm{SQL}}$, with ΔX_{SQL} the so-called 'standard quantum limit' for a conventional amplitude-and-phase measurement of X_1 or X_2, and so we can rewrite equation (35a) as

$$\langle\langle N\rangle\rangle \simeq \eta(\Delta X_{\mathrm{SQL}})^2 t. \tag{35b}$$

We have also seen in equation (27b) that the right-hand side of equation (35b) also gives at large times an upper bound to the root mean square variance in N,

$$E[\langle(\Delta N)^2\rangle]^{\frac{1}{2}} \leqslant \eta(\Delta X_{\mathrm{SQL}})^2 t. \tag{35c}$$

For the quantum nondemolition variables $X_{1,2}$, we have seen in equation (10) that the double expectation is not influenced by stochastic reduction or decoherence effects,

$$\delta\langle\langle X_{1,2}\rangle\rangle = 0, \tag{35d}$$

while from equation (27a) we get at large times an upper bound to the root mean square variances in $X_{1,2}$,

$$E[\langle(\Delta X_{1,2})^2\rangle]^{\frac{1}{2}} \leqslant (2\eta t)^{\frac{1}{2}}\sigma^2 = (2\eta t)^{\frac{1}{2}}(\Delta X_{\mathrm{SQL}})^2. \tag{36a}$$

This last equation has a similar form to the corresponding equation for a free particle, for which the standard quantum limit Δq_{SQL} in a position measurement is given [10] by

$$\Delta q_{\mathrm{SQL}} = (\hbar t/m)^{\frac{1}{2}}, \tag{36b}$$

so that at large times we have from equation (13c), and the fact that $\delta\,\mathrm{Tr}\,\rho q = 0$,

$$E[\langle(\Delta q)^2\rangle]^{\frac{1}{2}} \leqslant (\delta\,\mathrm{Tr}\,\rho q^2)^{\frac{1}{2}} = (\eta t/3)^{\frac{1}{2}}\hbar t/m = (\eta t/3)^{\frac{1}{2}}(\Delta q_{\mathrm{SQL}})^2. \qquad (36c)$$

These equations will form the basis for our analysis of experiments in which oscillating or free masses are monitored. Since we are making only order of magnitude estimates, we shall neglect numerical factors of order unity (such as the factor of 3 arising from generalizing from one to three dimensions), quoting all answers as powers of 10.

To make estimates, we shall need values of both the stochasticity parameter η and the elapsed time t. The value of η depends on the stochastic reduction model under consideration. In the GRW model [11] and also the QMUPL model [11], $\eta = \eta_0 N$, with $\eta_0 \sim 10^{-2}$ s^{-1} m^{-2} and with N the number of nucleons that are displaced in the measurement. For the CSL model, one has [8,11] $\eta = \gamma S^2 D^2 (\alpha/\pi)^{\frac{1}{2}}$, with S the side length (for a cube of material), D the density and $\gamma \sim 10^{-30}$ cm^3 s^{-1} and $\alpha \sim 10^{10}$ cm^{-2} parameters of the model. We shall assume a nucleon density of $D \sim 10^{24}$ cm^{-3}, and shall ignore the geometry dependence of η by eliminating S in terms of D and the nucleon number N by writing $S^2 = (N/D)^{\frac{2}{3}}$, giving

$$\eta = \gamma N^{\frac{2}{3}} D^{\frac{4}{3}} (\alpha/\pi)^{\frac{1}{2}}. \qquad (37a)$$

For the elapsed time t we shall take the inverse of the noise bandwidth frequency $F = \omega/(2Q)$, with Q the quality factor, for the nanomechanical resonator experiment, and the inverse of the low frequency limit of the sensitive range for the gravitational wave detector experiments. Our reasoning here is that if accumulation of a small stochastic effect takes longer than the time estimated this way, the effects will be hard to distinguish from accumulated effects of the noise that sets the low frequency limit of the detector.

The first experiment that we shall consider is the nanomechanical resonator reported by LaHaye *et al* [15], which uses a 19.7 MHz mechanical resonator containing $\sim 10^{12}$ nucleons, corresponding to $\Delta X_{\mathrm{SQL}} \sim 10^{-14}$ m, and which has a noise bandwidth $F = 903$ Hz. For the GRW and QMUPL models, we have $\eta \sim 10^{10}$ s^{-1} m^{-2}, giving an accumulated $\langle\langle N\rangle\rangle$ in time F^{-1} of 10^{-21}, and a root mean square expected deviation in $X_{1,2}$ of $\sim 10^{-10}\Delta X_{\mathrm{SQL}}$. In the CSL model, $\eta \sim 10^{19}$ s^{-1} m^{-2}, giving an accumulated $\langle\langle N\rangle\rangle$ in time F^{-1} of 10^{-12}, and a corresponding root mean square expected deviation in $X_{1,2}$ of $\sim 10^{-6}\Delta X_{\mathrm{SQL}}$.

The second experiment that we shall consider is the upgraded version of LIGO (the Advanced LIGO Interferometers), which monitors a quasi-free mass of 40 kg $\sim 10^{28}$ nucleons, and has a sensitive range extending down to $F \sim 70$ Hz. In the GRW model this apparatus has $\eta \sim 10^{27}$ s^{-1} m^{-2}, while in the CSL model the value of this parameter is $\eta \sim 10^{30}$ s^{-1} m^{-2}. The standard quantum limit of equation (36b) for position measurement over a time interval $t = (70\ \mathrm{Hz})^{-1}$ is $\Delta q_{\mathrm{SQL}} \sim 10^{-19}$ m, and we find that the root mean square stochastic deviation in the coordinate q over this time interval is bounded by $\sim 10^{-7}\Delta q_{\mathrm{SQL}}$ in the GRW model, and by $\sim 10^{-5}\Delta q_{\mathrm{SQL}}$ in the CSL model.

The third experiment that we consider is the projected space-based Laser Interferometer Space Antenna (LISA) [17], which will monitor the positions of 2 kg masses to an accuracy of 10^{-11} m, and which will be sensitive to frequencies down to 10^{-4} Hz. From equation (36b), the standard quantum limit corresponding to a 2 kg mass and $t \sim 10^4$ s is 10^{-15} m; in other words, this experiment will achieve a position accuracy of around 10^4 times Δq_{SQL}. For the CSL model, the corresponding root mean square stochastic deviation in the coordinate will be of order $100\Delta q_{\mathrm{SQL}}$, which is still a factor of 100 smaller than the observable displacement.

We see that in nanomechanical oscillator and advanced LIGO experiments, predicted stochastic reduction effects are at least a factor of 10^{-5} below the relevant standard quantum limits, and so are presently far from being detectable. The situation is better for LISA, where the stochastic reduction effect is predicted to be two orders of magnitude larger than the standard quantum limit, but still two orders of magnitude below the design position sensitivity. Even though these experiments are not expected to observe an effect, they will place useful bounds on the stochasticity parameter η. Trying to do better will be a challenging goal for future experiments; clearly, the key will be achieving a much larger accumulation time t, corresponding to a greatly reduced noise bandwidth F for the nanomechanical resonator, or a greatly reduced lower frequency limit for the gravitational wave detectors. We note in closing that when equation (5b) is used as a model for environmentally induced (as opposed to postulated intrinsic) decoherence effects, the appropriate value of η may be much larger than in the above estimates, and so in this case the effects for which we have obtained theoretical formulae may lie within reach of current experimental technique.

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Appendix A. Basic Itô calculus formulae

The stochastic differential $\mathrm{d}W_t$ behaves heuristically as a random square root of $\mathrm{d}t$, as expressed in the Itô calculus rules

$$\mathrm{d}W_t^2 = \mathrm{d}t, \qquad \mathrm{d}W_t\,\mathrm{d}t = \mathrm{d}t^2 = 0. \tag{A.1}$$

As a consequence of equation (A1), the Leibniz chain rule of the usual calculus is modified to

$$\mathrm{d}(AB) = (\mathrm{d}A)B + A\,\mathrm{d}B + \mathrm{d}A\,\mathrm{d}B. \tag{A.2}$$

Applying these two formulae to the definition

$$\hat{\rho}(t) = |\psi_t\rangle\langle\psi_t|, \tag{A.3}$$

the stochastic equation of motion of equation (4b) for $|\psi_t\rangle$ is easily seen to imply the equation of motion of equation (5a) for $\hat{\rho}(t)$. Because the Itô differential $\mathrm{d}W_t$ is statistically independent of the variables at and before time t, the final term in equation (5a) vanishes when the stochastic expectation $E[\hat{\rho}]$ is taken, giving equation (5b). Using this statistical independence of $\mathrm{d}W_t$, and equations (A1)–(A3), we can get a useful formula for the expectation of a product of stochastic integrals. Consider the expectation

$$f(t) = E\left[\int_0^t \mathrm{d}W_u A(u) \int_0^t \mathrm{d}W_u B(u)\right], \tag{A.4}$$

which by equation (A2) has the differential

$$\mathrm{d}f(t) = E\left[\mathrm{d}W_t A(t) \int_0^t \mathrm{d}W_u B(u) + \int_0^t \mathrm{d}W_u A(u)\,\mathrm{d}W_t B(t) + A(t)B(t)\,\mathrm{d}t\right] = E[A(t)B(t)]\,\mathrm{d}t. \tag{A.5}$$

Integrating back using the right hand-side of equation (A5), we get

$$E\left[\int_0^t \mathrm{d}W_u A(u) \int_0^t \mathrm{d}W_u B(u)\right] = \int_0^t \mathrm{d}u\, E[A(u)B(u)], \tag{A.6}$$

a formula called the Itô isometry.

Appendix B. Connection to the Lindblad evolution equation

The most general completely positive density matrix evolution equation is given by the form studied by Lindblad [18] and Gorini, Kossakowski and Sudarshan [18], generally referred to as the Lindblad equation,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H,\rho] + \sum_j \left(L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\} \right). \tag{B.1}$$

When L_j is self-adjoint, so that $L_j = L_j^\dagger$, the summand in equation (B1) reduces to the form

$$-\tfrac{1}{2}[L_j, [L_j, \rho]], \tag{B.2}$$

which corresponds to the decoherence equation studied in the text when we take $L_j = q$, and more generally leads to a solvable oscillator model when $L_J = c_1 q + c_2 p$ (with self-adjointness requiring real $c_{1,2}$). These two cases correspond respectively to repeated environmental (or intrinsic, in the case of reduction models) measurements of the system coupling to q or to $c_1 q + c_2 p$.

Dissipative equations in the Lindblad context are generated by taking L_j to be non-self-adjoint. For example, if we take $L_j = a$ for the harmonic oscillator, we get additional terms that cannot be represented as double commutators, as seen from the identity

$$4\sigma^2 \left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\} \right] = -\frac{1}{2}[q,[q,\rho]] - \frac{1}{2m^2\omega^2}[p,[p,\rho]] - \frac{\mathrm{i}}{2m\omega}([q,\{p,\rho\}] - [p,\{q,\rho\}]), \tag{B.3}$$

and a similar identity in which a is interchanged with $a^\dagger$ and the 'i' on the right-hand side is replaced by $-\mathrm{i}$. The paper of Salama and Gisin [14] studies a dissipative Lindblad equation with a term $L_j \propto a$, while the papers of Isar, Sandulescu and Scheid [14] and Karrlein and Grabert [14] study a non-Lindblad master equation with a single dissipative term proportional to $[q, \{p, \rho\}]$.

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